Love-for-Variety

Kiminori Matsuyama
Northwestern University

Philip Ushchev
ECARES, Université Libre de Bruxelles

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Some Backgrounds

“Non-CES Aggregators: A Guided Tour”
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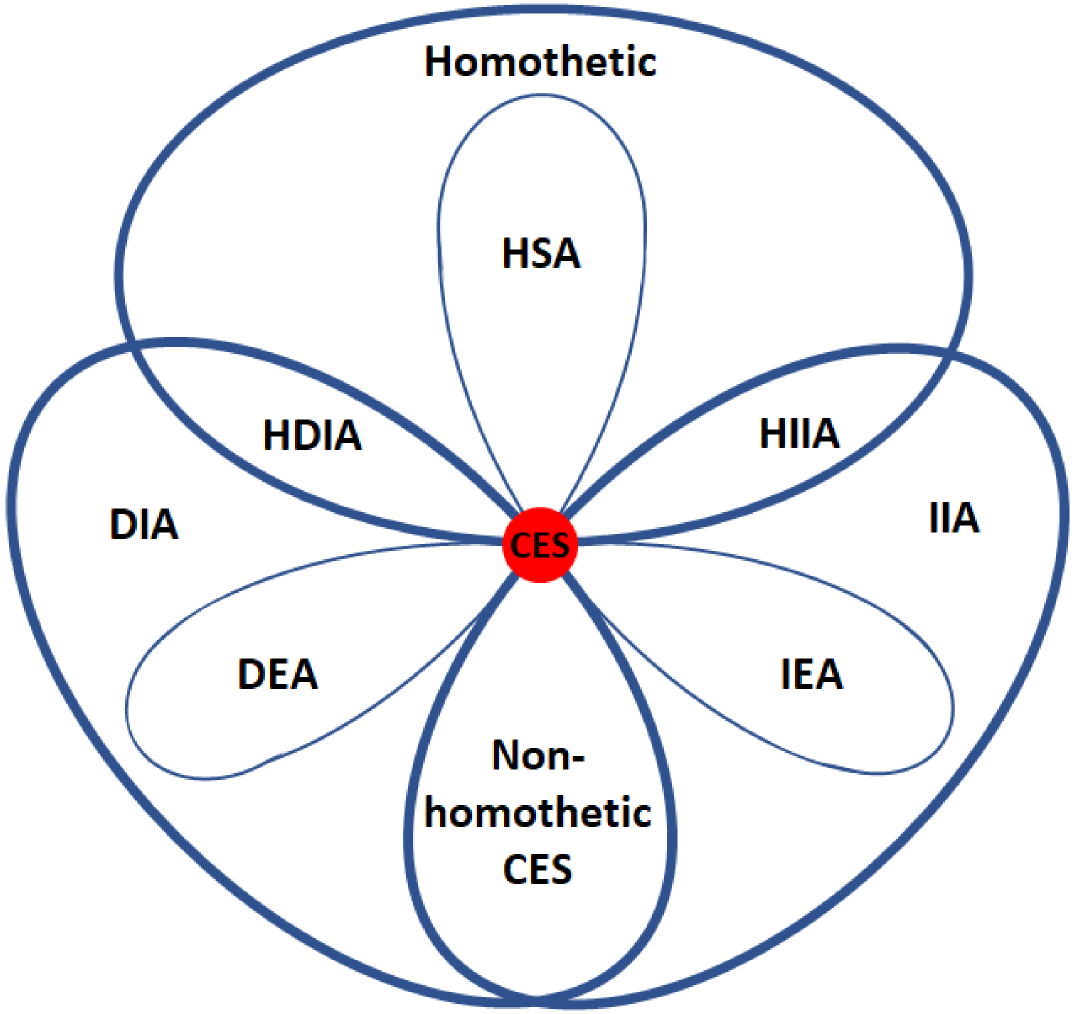
- We all love using CES, because it is tractable.
- CES is tractable because it has many knife-edge properties, which also make it restrictive.
- For some purposes, we need to drop some properties.
- Many look for an alternative, such as Stone-Geary, translog, etc. But they have their own drawbacks.
- My Approach: Relax only those properties we need to drop and keep the rest to retain the tractability of CES as much as possible.
- Depending on which properties are kept, we come up with many different classes of non-CES demand systems.
- Which class should be used depends on the applications.



Love-for-Variety

Introduction

Love-for-Variety (LV): Productivity (utility) gains from increasing variety of intermediate inputs (consumer goods).

- A natural consequence of the convexity of the production technologies (preferences).
- Willingness to pay for new inputs (goods); Dixit-Stiglitz (1977), Krugman (1980), Ethier (1982), Romer (1987), etc.
- A central concept in economic growth (Grossman-Helpman 1993; Gancia-Zillibotti 2005, Acemoglu 2008), international trade (Helpman-Krugman 1995), and economic geography (Fujita-Krugman-Venables 1999).
- Though commonly discussed in monopolistic competition settings, LV is also a useful concept in other contexts, such as gains from trade in Armington-type competitive models.

Little is known about how LV depends on the underlying demand system outside of **CES with gross substitutes:**

- The LV measure under CES: $\mathcal{L} = 1/(\sigma - 1) > 0,$ where $\sigma > 1$ *happens to* represent 2 related but distinct concepts,
    - the elasticity of substitution (ES) across varieties.
    - the price elasticity (PE) of demand for each variety.
  - ✓ **One appealing feature:** LV is smaller when ES is larger and when PE is larger.
  - ✓ **One unappealing feature:** LV is independent of how many varieties are already available.

For this reason, some may prefer "Ideal variety approach," but it is less tractable than "Love-for-variety approach."

The Questions: What happens outside of CES?

- How is LV related to the underlying demand structure, such as ES or PE?

Note: ES and PE are distinct concepts, which could play different roles shaping LV outside of CES.

- Under what conditions does LV decline as more varieties become available?

- Can we develop "Love-for-variety approach" with diminishing LV, which is also tractable?

Our Approach to These Questions

- Define **Substitutability,** $\sigma(V)$, & **Love-for-Variety,** $\mathcal{L}(V)$; both depend only on V (the mass of available varieties).
    - Under CES, there are independent of V, as $\sigma(V) = \sigma$; $\mathcal{L}(V) = 1/(\sigma - 1)$.

- One's intuition might say that **Increasing Substitutability** implies **Diminishing love-for-variety.**

$$\sigma'(V) > 0 \Longrightarrow \mathcal{L}'(V) < 0.$$

- It turns out that this is NOT true under **general symmetric homothetic demand systems.** Little can be said about the relations btw PE, $\sigma(V)$ & $\mathcal{L}(V)$. "Almost anything goes."

- To capture the above intuition, we need to impose more restrictions. Homotheticity (and symmetry) just too broad.

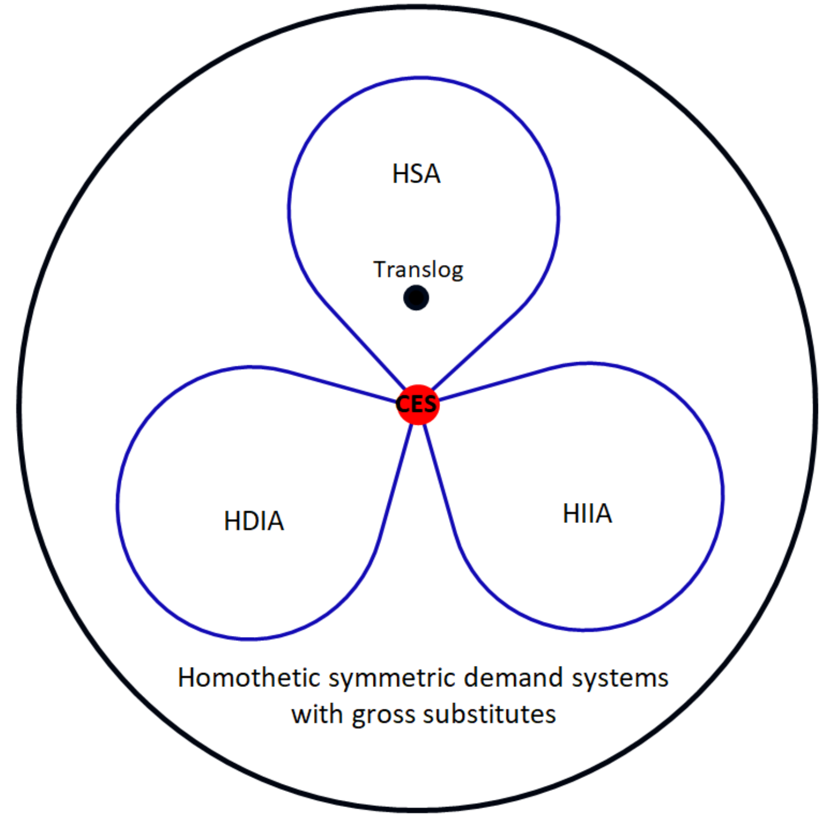
We turn to the **3 classes of homothetic demand systems:**

H.S.A. (Homothetic Single Aggregator)
HDIA (Homothetic Direct Implicit Additivity)
HIIA (Homothetic Indirect Implicit Additivity)

- Pairwise disjoint with the sole exception of CES.
- PE can be written as $\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right) \equiv \zeta^*\left(\frac{x_\omega}{\mathcal{A}^*(\mathbf{x})}\right)$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is linear homogeneous, a sufficient statistic for the cross-variety effects.

Main Results: In each of these 3 classes,

- $\sigma'(V) > 0 \Leftrightarrow$ The 2nd law.
- $\sigma'(V) \gtreqless 0 \Rightarrow \mathcal{L}'(V) \lesseqgtr 0$. The converse is not true.
- $\mathcal{L}'(V) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \zeta(\cdot) = \zeta^*(\cdot) = const.$, which occur iff CES.



The 3 classes offer a tractable way of capturing the intuition that **gains from increasing variety is diminishing, if varieties are more substitutable in the presence of more varieties.**

The 3 classes also are useful as building blocks for more general (but not fully general) demand systems.

Some Remarks Before Proceeding,

- **This paper is all about the *demand side* of LV**, hence applicable to a wide range of models.

- **We deliberately make no assumption on the supply side.** For example,

  - **Armington-type models**, where each differentiated input (or consumer good) is produced and sold by competitive producers, and the mass of available varieties, V, changes exogenously due to trade liberalization
  - **Central planning problems**, where the benevolent planner chooses V optimally subject to the innovation cost.
  - **Oligopoly models** with a finite number of oligopolistic firms, each of which innovate and produce a continuum range of varieties.
  - **Monopolistically competitive models**, with a continuum of monopolistically competitive firms innovating and producing zero measure of varieties and selling them with positive markups.

- **Neither symmetry nor homotheticity are as restrictive as they look.**

  - By nesting symmetric homothetic demand systems into a upper-tier asymmetric/nonhomothetic demand system, we can create an asymmetric/nonhomothetic demand system.
  - Moreover, one key message is "Almost anything goes," that symmetry/homotheticity restrictions are *not restrictive enough* that we need to look for more restrictions to make further progress.

General Symmetric Homothetic Demand Systems

General Symmetric Homothetic (Input) Demand System: A Quick Refresher of Duality Theory

Consider homothetic demand systems for a continuum of differentiated inputs generated by symmetric CRS production technology.

CRS Production Function	**Unit Cost Function**
$X(\mathbf{x}) \equiv \min_{\mathbf{p}}\{\mathbf{px} \mid P(\mathbf{p}) \geq 1\}$	$P(\mathbf{p}) \equiv \min_{\mathbf{x}}\{\mathbf{px} \mid X(\mathbf{x}) \geq 1\}$

$\mathbf{x} = \{x_\omega; \omega \in \bar{\Omega}\}$**:** the input quantity vector; $\mathbf{p} = \{p_\omega; \omega \in \bar{\Omega}\}$**:** the input price vector.

$\bar{\Omega}$, a continuum of all potential input varieties. $\Omega \subset \bar{\Omega}$, the set of available input varieties, with its mass $V \equiv |\Omega|$.

$\bar{\Omega} \backslash \Omega$: the set of unavailable varieties, $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \bar{\Omega} \backslash \Omega$.

Inputs are *inessential*, i.e., $\bar{\Omega} \backslash \Omega \neq \emptyset$ does*n't* imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$.

Duality Principle:

Either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be a *primitive*, if linear homogeneity, monotonicity & strict quasi-concavity satisfied

Demand System:

Demand Curve (from Shepherd's Lemma)	**Inverse Demand Curve**
$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x})$	$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}$

And, from Euler's Homogenous Function Theorem,

$$\mathbf{px} = \int_\Omega p_\omega x_\omega d\omega = \int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) d\omega = \int_\Omega P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega = P(\mathbf{p}) X(\mathbf{x}) = E.$$

The value of inputs is equal to the total value of output under CRS.

Budget Share of $\omega \in \Omega$:	$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega, \mathbf{x})$

Under general CRS, little restrictions on s_ω beyond homogeneity of degree zero in $(p_\omega, \mathbf{p})$ or in $(x_\omega, \mathbf{x})$. **→** $s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega)$, depends on the *distribution* of the prices (quantities) divided by its own price (quantity).

Definition: Gross Substitutability

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} > 0$$

Price Elasticity of Demand for $\omega \in \Omega$	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) \equiv 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega; \mathbf{x}) \equiv \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} > 1.$

Under general CRS, little restrictions on ζ_ω, beyond the homogeneity of degree zero in $(p_\omega, \mathbf{p})$ or in $(x_\omega, \mathbf{x})$. **→** $\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega)$, depends on the whole *distribution* of prices (quantities) divided by its own price (quantity).

Definition: The 2nd Law of Demand

$$\frac{\partial \ln \zeta(p_\omega; \mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} < 0.$$

Clearly, CES does not satisfy the 2nd Law.

Substitutability Measure Across Different Varieties

Unit Quantity Vector: $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \bar{\Omega}\}$, where $(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Unit Price Vector: $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \bar{\Omega}\right\}$, where $\left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Note: $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$.

At the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$,

$$s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega) = s\left(1, \mathbf{1}_\Omega^{-1}\right) = s^*(1, \mathbf{1}_\Omega) = 1/V$$

$$\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega) = \zeta\left(1, \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1, \mathbf{1}_\Omega) > 1$$

Clearly, this depends only on V. We propose:

Definition: *The substitutability measure across varieties* is defined by

$$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega) > 1.$$

We call the case of $\sigma'(V) > (<) 0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

Alternatively, we can define the substitutability by the Allen-Uzawa elasticity of substitution btw ω and ω', $AES(p_\omega, p_{\omega'}, \mathbf{p})$, at the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$. It turns out that these definitions are equivalent because

$$\sigma(V) = AES\left(p, p, p\mathbf{1}_\Omega^{-1}\right) = AES\left(1,1, \mathbf{1}_\Omega^{-1}\right).$$

Note: In general, the 2nd Law is neither sufficient nor necessary for increasing substitutability, $\sigma'(V) > 0$.

Love-for-Variety Measure: Commonly defined by the productivity gain from a higher V, holding xV

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0$$

Alternatively, it may be defined by the decline in $P(\mathbf{p})$ from a higher V, at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, holding p constant.

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} > 0.$$

Both are functions of V only, and equivalent because, by applying $\mathbf{x} = x\mathbf{1}_\Omega$ and $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ to $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$,

$$pxV = pP\left(\mathbf{1}_\Omega^{-1}\right)xX(\mathbf{1}_\Omega) \Longrightarrow -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv \frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Note: $\mathcal{L}(V) > 0$ is guaranteed by the strict quasi-concavity.

Example: Standard CES with Gross Substitutes:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_\omega^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \iff P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$. ($Z > 0$ is TFP or affinity in the preference, in the context of spatial economics)

	Definition	**Under CES**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$

Under Standard CES,

- Price elasticity of demand, $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, is independent of $\mathbf{p}$ or $\mathbf{x}$ and equal to σ.
- Substitutability, $\sigma(V)$, is independent of V and equal to σ.
- Love-for-variety, $\mathcal{L}(V)$, is independent of V, and equal to a constant, $\mathcal{L}(V) = \mathcal{L} = 1/(\sigma - 1)$, inversely related to σ.

These properties do not hold under general homothetic demand systems.

Example: Generalized CES with Gross Substitutes a la Benassy (1996).

$$X(\mathbf{x}) = Z(V)\left[\int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_\Omega {p_\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

Note: $Z(V)$ allows variety to have direct externalities to TFP (or affinity)

	Definition	**Under Generalized CES**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{d \ln Z(V)}{d \ln V}.$

Under Generalized CES,

- Price Elasticity, $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, and Substitutability, $\sigma(V)$, are not affected by $\frac{d \ln Z(V)}{d \ln V}$.
- $\frac{d \ln Z(V)}{d \ln V}$, the Benassy residual, "accounts for" the discrepancy between the LV implied by CES and the observed LV.
  - Benassy (1996) set $\frac{d \ln Z(V)}{d \ln V} = \nu - \frac{1}{\sigma-1}$, so that $\mathcal{L}(V) = \nu$ is a separate parameter independent of σ.
  - If we instead assume that $\frac{d \ln Z(V)}{d \ln V}$ is independent of σ, $\mathcal{L}(V)$ is still inversely related to σ.

Even if you believe in the Benassy residual, your estimate of its magnitude depends on the CES structure.

General Homothetic Demand System: The relation btw $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, $\sigma(V)$, & $\mathcal{L}(V)$ can be complex.

- Whether Marshall's 2nd Law holds or not says little about the derivatives of $\sigma(V)$ and $\mathcal{L}(V)$.
- $\sigma(V)$ and $\mathcal{L}(V)$ could be positively related.

(Counter)Example: Weighted Geometric Mean of Standard Synmetric CES with Gross Substitutes:

$$X(\mathbf{x}) \equiv \exp\left[\int_1^\infty \ln X(\mathbf{x};\sigma)\, dF(\sigma)\right], \qquad \text{where} \qquad [X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega$$

and $F(\cdot)$ is a c.d.f. of $\sigma \in (1,\infty)$, satisfying $\int_1^\infty dF(\sigma) = 1$.

	Definition	Under Weighted Geometric Mean of CES
Price Elasticity	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta^*(x_\omega; \mathbf{x})$	$\zeta^*(x_\omega; \mathbf{x}) = E_F\left((x_\omega)^{-\frac{1}{\sigma}} \big/ \left(X(\mathbf{x};\sigma)\right)^{1-\frac{1}{\sigma}}\right) \Big/ E_F\left((x_\omega)^{-\frac{1}{\sigma}} \big/ \sigma\left(X(\mathbf{x};\sigma)\right)^{1-\frac{1}{\sigma}}\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \dfrac{1}{E_F(1/\sigma)} > 1$
Love-for-variety	$\mathcal{L}(V) \equiv -\dfrac{d \ln \mathcal{Y}(V)}{d \ln V} - 1 > 0$	$\mathcal{L}(V) = E_F\left(\dfrac{1}{\sigma - 1}\right) > 0$

- PE, $\zeta^*(x_\omega; \mathbf{x})$, is not constant, and *violates* the Marshall's 2nd Law.
- $\sigma(V)$ and $\mathcal{L}(V)$ are both constant, *independent* of V.
- The range of $\sigma(V)$ and $\mathcal{L}(V)$ is $0 < \frac{1}{\sigma(V)-1} \leq \mathcal{L}(V) < \infty$, where the equality holds iff F is degenerate.
- Easy to construct a parametric family of F, such that $\sigma(V)$ and $\mathcal{L}(V)$ are positively related.

Three Classes of Symmetric Homothetic Demand Systems

However, it is intuitive to think that, as input varieties are more substitutable,

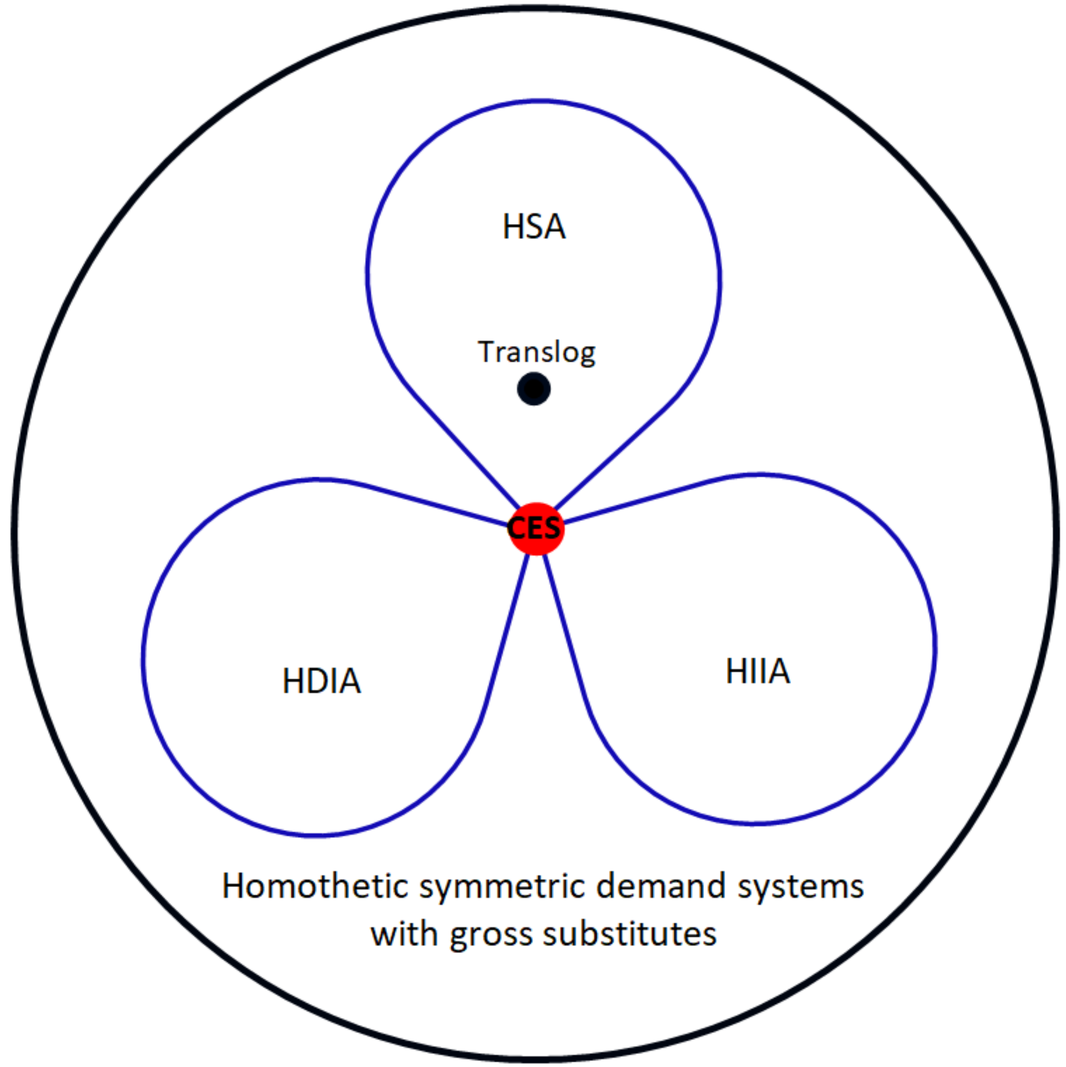
- the price elasticity of demand for each variety become larger,
- the love-for-variety measure become smaller.

Homotheticity is too general to capture this intuition!!
It is NOT restrictive enough.

In search for additional restrictions to capture this intuition, we turn to

Three Classes of Symmetric CRS Production Functions:

✓ **Homothetic Single Aggregator (H.S.A.)**
✓ **Homothetic Direct Implicit Additivity (HDIA)**
✓ **Homothetic Indirect Implicit Additivity (HIIA)**



3 Classes of Symmetric CRS Production Functions (with Gross Substitutes & Inessentiality)

Homothetic Direct Implicit Additivity (HDIA):

$$\int_{\Omega} \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega \equiv \int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1$$

$\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, TFP.
$\phi(0) = 0; \phi(\infty) = \infty; \phi'(y) > 0 > \phi''(y), -y\phi''(y)/\phi'(y) < 1$, for $\forall y \in (0, \infty)$.
CES with $\phi(y) = (y)^{1-1/\sigma}$.

Homothetic Indirect Implicit Additivity (HIIA):

$$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega \equiv \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega \equiv 1$$

$\theta(\cdot): \mathbb{R}_{++} \to \mathbb{R}_+$ is independent of $Z > 0$ is TFP.
$\theta(z) > 0, \theta'(z) < 0 < \theta''(z), -z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \leq \infty$ & $\theta(z) = 0$ for $z \geq \bar{z}$.
CES with $\theta(z) = (z)^{1-\sigma}$.

Homothetic Single Aggregator (H.S.A.): Two Equivalent Definitions

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \quad \text{with} \quad \int_{\Omega} s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1 \quad \Leftrightarrow \quad s_\omega = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \quad \text{with} \quad \int_{\Omega} s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1$$

$s(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, TFP.
$s(z) > 0, s'(z) < 0$ for $0 < z < \bar{z} \leq \infty$; $s(z) = 0$ for $z \geq \bar{z}$.
CES with $s(z) = \gamma z^{1-\sigma}$.

$s^*(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, TFP.
$s^*(0) = 0, s^*(y) > 0, \ 0 < ys^{*\prime}(y)/s^*(y) < 1$.
CES with $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$.

The definition of H.S.A. is independent of $Z > 0$, TFP, which shows up when we integrate the definition to obtain $P(\mathbf{p})$ or $X(\mathbf{x})$.

Key Properties of the Three Classes

	Budget Shares: $s_\omega \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(p_\omega; \mathbf{p})$		Price Elasticity: $\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p})$
CES	$s_\omega \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = f\left(\frac{p_\omega}{P(\mathbf{p})}\right) \Leftrightarrow s_\omega \propto \left(\frac{p_\omega}{P(\mathbf{p})}\right)^{1-\sigma}$		σ
H.S.A.	$s_\omega = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{A(\mathbf{p})} \neq c$, unless CES	$\zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right)$; $\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1$
HDIA	$s_\omega = \frac{p_\omega}{P(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{B(\mathbf{p})} \neq c$, unless CES	$\zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$; $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$
HIIA	$s_\omega = \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{C(\mathbf{p})} \neq c$, unless CES	$\zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$; $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1.$

$A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})$: all linear homogenous, determined implicitly by the adding-up constraint, $\int_\Omega s_\omega d\omega \equiv 1$.

We focus on these three classes for two reasons.

- They are pairwise disjoint with the sole exception of CES.
- Price elasticity can be written as $\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right) \equiv \zeta^*\left(\frac{x_\omega}{\mathcal{A}^*(\mathbf{x})}\right)$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is linear homogenous, a sufficient statistic, which captures all the cross-variety effects.

Key Properties of the Three Classes, continued.

	CES	H.S.A.	HDIA	HIIA
Price Elasticity $\zeta(p_\omega;\mathbf{p})$	σ	$\zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right)$; $\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1$	$\zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$ $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$	$\zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$; $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1$
Substitutability $\sigma(V)$	σ	$\zeta\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$
Love-for-Variety $\mathcal{L}(V)$	$\frac{1}{\sigma-1}$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$; $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi > 0$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$; $0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1$	$\frac{1}{\mathcal{E}_\theta(\theta^{-1}(1/V))}$; $\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0$

Main Results: In each of these 3 classes, Under H.S.A., HDIA, and HIIA,

- $\sigma'(V) > 0$ iff the 2nd law holds.
- $\sigma'(V) \gtreqless 0$ for all $V > 0 \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0$ for all $V > 0$. The converse is not true. But,
- $\mathcal{L}'(V) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \zeta(\cdot) = \zeta^*(\cdot) = const.$, which occur iff CES.

Homothetic Single Aggregator (H.S.A.)

Symmetric H.S.A. (Homothetic Single Aggregator) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $z_\omega \equiv p_\omega / A$, its own price p_ω, normalized by the common price aggregator, $A = A(\mathbf{p})$.

$$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \qquad \text{where} \qquad \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_+$: **the budget share function**, decreasing in the normalized price, $z_\omega \equiv p_\omega / A$ for $s(z_\omega) > 0$ with
  - $\lim_{z \to \bar{z}} s(z) = 0$. If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(\mathbf{p})$ is **the choke price.**
- $A = A(\mathbf{p})$: **the common price aggregator**, defined implicitly by **the adding-up constraint**, $\int_\Omega s(p_\omega / A) d\omega \equiv 1$.

By construction, the budget shares add up to one. $A(\mathbf{p})$ linear homogenous in $\mathbf{p}$ for a fixed Ω. A larger Ω reduces $A(\mathbf{p})$.

Some Special Cases

CES with gross substitutes	$s(z) = \gamma z^{1-\sigma}$;	$\sigma > 1$
Translog Cost Function	$s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}$;	$\bar{z} < \infty$
Constant Pass Through (CoPaTh)	$s(z) = \gamma \max\left\{\left[\sigma - (\sigma - 1) z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}, 0\right\}$	$\sigma > 1;\ 0 < \rho < 1$

As $\rho \nearrow 1$, CoPaTh converges to CES with $\bar{z} = \left(\frac{\sigma}{\sigma - 1}\right)^{\frac{\rho}{1-\rho}} \to \infty$.

Price Elasticity:	$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta(z_\omega) > 1$

Notes:

- A function of a single variable, $z_\omega \equiv p_\omega / A(\mathbf{p})$.
- $\zeta(z_\omega) = \sigma > 1$ under CES, $s(z) = \gamma z^{1-\sigma}$.
- Marshall's 2nd law iff $\zeta'(\cdot) > 0$, e.g., $\zeta(z_\omega) = 1 - \frac{1}{\ln(z_\omega/\bar{z})}$ for translog; $= \frac{\sigma}{\sigma-(\sigma-1){z_\omega}^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ for CoPaTh.

Unit Cost Function: By integrating $\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$,

$$\frac{A(\mathbf{p})}{P(\mathbf{p})} = c \exp\left[\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right)\Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega\right], \quad \text{where} \quad \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0.$$

where $c > 0$ is a constant, proportional to TFP.

Notes:

- $P(\mathbf{p})$: linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$, with the sole exception of CES, because

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'}) z_{\omega'} d\omega'} = \frac{[\zeta(z_\omega) - 1] s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'}) - 1] s(z_{\omega'}) d\omega'} \neq \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta(z)$ is independent of z or $s(z) = \gamma z^{1-\sigma}$ with $\zeta(z) = \sigma > 1$.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$1 = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)V = s\left(\frac{p}{A(p\mathbf{1}_\Omega^{-1})}\right)V = s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right)V \Rightarrow z_\omega = \frac{p_\omega}{A(\mathbf{p})} = \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	Under H.S.A.
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta\left(s^{-1}(1/V)\right) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) > 0.$

Notes:

- At symmetric price patterns,

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \ln\left[\frac{A\left(\mathbf{1}_\Omega^{-1}\right)}{cP\left(\mathbf{1}_\Omega^{-1}\right)}\right] = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \mathcal{L}(V).$$

- Since $s^{-1}(1/V)$ is increasing in V,

$$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

implies that Marshall's 2nd law, $\zeta'(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under H.S.A.

$$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right); \ \mathcal{L}(V) = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right), \qquad \text{where} \qquad \zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)}; \ \Phi(z) \equiv \frac{1}{s(z)}\int_{z}^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi.$$

Lemma 1:

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$

Furthermore,

$$\zeta'(z) = 0 \iff \Phi'(z) = 0 \iff \text{CES}.$$

From this,

Proposition 1

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/s(z_0), \infty)$$

$$\implies$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/s(z_0), \infty).$$

Furthermore,

$$\zeta'(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \Phi'(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

Thus, under H.S.A.,

- Marshall's 2nd Law, $\zeta'(\cdot) > 0$ for all $z < \bar{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Homothetic Direct Implicit Additivity (HDIA)

Symmetric HDIA (Homothetic Directly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ is called *homothetic with direct implicit additivity* (HDIA) with gross substitutes if it can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1,$$

where $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, C^3, with $\phi(0) = 0; \phi(\infty) = \infty; \phi'(y) > 0 > \phi''(y)$, $-y\phi''(y)/\phi'(y) < 1, \forall y \in (0,\infty)$.

- By construction, $\hat{X}(\mathbf{x})$ is independent of $Z > 0$, TFP.
- If $\phi'(0) < \infty$, the choke price is $B(\mathbf{p})\phi'(0)$. If $\phi'(0) = \infty$, no choke price.
- CES with gross substitutes: $\phi(y) = (y)^{1-1/\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\phi(y) = \int_0^{y}\left(1 + \frac{1}{\sigma-1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi$, $0 < \rho < 1$, converging to CES with $\rho \nearrow 1$.
- An extension of the Kimball (1995) aggregator in the sense that Ω is not fixed and $V \equiv |\Omega|$ is a variable.

Inverse Demand Curve:	$\frac{p_{\omega}}{B(\mathbf{p})} = \phi'\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right) = \phi'\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{Zx_{\omega}}{X(\mathbf{x})} = \frac{x_{\omega}}{\hat{X}(\mathbf{x})} = (\phi')^{-1}\left(\frac{p_{\omega}}{B(\mathbf{p})}\right)$
Unit Cost Function:	$P(\mathbf{p}) = \frac{1}{Z}\hat{P}(\mathbf{p}) \equiv \frac{1}{Z}\int_{\Omega} p_{\omega}(\phi')^{-1}\left(\frac{p_{\omega}}{B(\mathbf{p})}\right) d\omega$		

where $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \phi\left((\phi')^{-1}\left(\frac{p_{\omega}}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \frac{x_\omega}{C^*(\boldsymbol{x})}\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right),$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega x_\omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega$$

satisfying the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}.$$

Budget share under HDIA: A function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ & $p_\omega/B(\mathbf{p})$, or of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ & $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$\zeta_\omega = \zeta^*(x_\omega;\mathbf{x}) = -\frac{\phi'(y_\omega)}{y_\omega \phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) = \zeta(p_\omega;\mathbf{p}) > 1$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ or $\phi'(y_\omega) = p_\omega/B(\mathbf{p})$.
- $\zeta^D(y_\omega) = \sigma > 1$ under CES, $\phi(y) = (y)^{1-1/\sigma}$
- Marshall's 2nd law iff $\zeta^{D'}(\cdot) < 0$, satisfied by $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$ under CoPaTh.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \implies \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	**Under HDIA**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega = \zeta^D\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) > 1,$
Substitutability	$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^D(\phi^{-1}(1/V)) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln y(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1 > 0.$

where

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$

Notes:

- At symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \implies \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{X}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} = \mathcal{L}(V) + 1.$$

- Since $\phi^{-1}(1/V)$ is decreasing in V,

$$\sigma(V) = \zeta^D\left(\phi^{-1}(1/V)\right)$$

implies that Marshall's 2nd law, $\zeta^{D\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HDIA.

$$\sigma(V) = \zeta^D\left(\phi^{-1}(1/V)\right);\ \mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V)\,)} - 1, \qquad \text{where} \qquad \zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)};\ \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)}$$

Hence,

Lemma 2:

$$\zeta^{D'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \mathcal{E}'_\phi(y) = 0 \iff \text{CES}.$$

From this,

Proposition 2:

$$\zeta^{D'}(y) \lesseqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\phi(y_0)\,, \infty)$$

$$\implies$$

$$\mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\phi(y_0)\,, \infty).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \sigma'(V) = 0 \Leftrightarrow \mathcal{E}'_\phi(y) = 0 \iff \mathcal{L}'(V) = 0 \iff \text{CES}.$$

Thus, under HDIA,

- Marshall's 2nd Law, $\zeta^{D\prime}(\cdot) < 0$ for all $y > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Homothetic Indirect Implicit Additivity (HIIA)

Symmetric HIIA (Homothetic Indirectly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, is called *homothetic with indirect implicit additivity* (HIIA) if it can be defined implicitly by:

$$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1,$$

where $\theta: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is independent of $Z > 0, C^3$, with $\theta(z) > 0, \theta'(z) < 0, \theta''(z) > 0, -z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0}\theta(z) = \infty$ and $\lim_{z\to\bar{z}}\theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$.

- By construction, $\hat{P}(\mathbf{p})$ is independent of $Z > 0$, TFP.
- If $\bar{z} < \infty$, $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$ is the choke price. If $\bar{z} = \infty$, no choke price.
- CES with gross substitutes: $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\theta(z) = \sigma^{\frac{\rho}{1-\rho}} \int_{z/\bar{z}}^{1} \left((\xi)^{\frac{\rho-1}{\rho}} - 1\right)^{\frac{\rho}{1-\rho}} d\xi$ for $z < \bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$; $0 < \rho < 1$, converging to CES as $\rho \nearrow 1$.

Inverse Demand Curve:	$\frac{p_\omega}{ZP(\mathbf{p})} = \frac{p_\omega}{\hat{P}(\mathbf{p})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{x_\omega}{B^*(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = -\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) > 0$
Production Function:	$X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z\int_{\Omega} (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega$		

where $\hat{X}(\mathbf{x})$ and $B^*(\mathbf{x})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\boldsymbol{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\frac{p_\omega}{C(\mathbf{p})}$

where

$$C(\mathbf{p}) \equiv -\int_\Omega \theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)p_\omega d\omega > 0$$

satisfying the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})}d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}.$$

Budget share under HIIA: A function of two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or of two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$\zeta_\omega = \zeta(p_\omega;\mathbf{p}) = -\frac{z_\omega\theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) = \zeta^*(x_\omega;\ \mathbf{x}) > 1$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $z_\omega \equiv p_\omega/\hat{P}(\mathbf{p})$ or $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$.
- $\zeta^I(z_\omega) = \sigma > 1$ under CES, $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- Marshall's 2nd law iff $\zeta^{I\prime}(z_\omega) > 0$, satisfied by $\zeta^I(z_\omega) = \frac{\sigma}{\sigma-(\sigma-1)(z_\omega)^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ under CoPaTh.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \implies \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}(1/V).$$

Hence,

	Definition	**Under HIIA**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^I\big(\theta^{-1}(1/V)\big) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln y(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta\big(\theta^{-1}(1/V)\big)} > 0.$

where

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$

Notes:

- At symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\frac{C(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \implies \mathcal{L}(V) = \frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{C(\mathbf{1}_\Omega^{-1})} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}$$

- Since $\theta^{-1}(1/V)$ is increasing in V,

$$\sigma(V) = \zeta^I\big(\theta^{-1}(1/V)\big)$$

implies that Marshall's 2nd law, $\zeta^{I\prime}(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HIIA.

$$\sigma(V) = \zeta^I\big(\theta^{-1}(1/V)\big);\ \mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta\big(\theta^{-1}(1/V)\big)}, \qquad \text{where} \qquad \zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)};\ \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)}.$$

Hence,

Lemma 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \quad \Longrightarrow \quad \mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \bar{z}).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \quad \Leftrightarrow \quad \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \text{CES}.$$

From this,

Proposition 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\theta(z_0), \infty)$$

$$\Longrightarrow$$

$$\mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\theta(z_0), \infty).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

Under HIIA,

- Marshall's 2nd Law, $\zeta^{I'}(\cdot) < 0$ for all $z < \bar{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Appendices

Appendix C: An Alternative (and Equivalent) Definition of H.S.A.

Definition: A symmetric CRS technology, X= $X(\mathbf{x})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $y_\omega \equiv x_\omega/A^*$, its own quantity x_ω, normalized by the common quantity aggregator, $A^* = A^*(\mathbf{x})$. $s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)$, where $\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1.$

- $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$: the *budget share function,* in $y_\omega \equiv x_\omega/A^*$ with $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1$, $s^*(0) = 0$, $s^*(\infty) = \infty$.
- $A^* = A^*(\mathbf{x})$: **the common quantity aggregator**, defined by **the adding-up constraint,** $\int_\Omega s^*(x_\omega/A^*)d\omega \equiv 1$. By construction, the budget shares add up to one. $A^*(\mathbf{x})$ linear homogenous in $\mathbf{x}$ for a fixed Ω. A larger Ω increases A^*.

Price Elasticity:	$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{d\ln s^*(y_\omega)}{d\ln y_\omega}\right]^{-1} \equiv \zeta^*(y_\omega) > 1,$

Notes:

- Also a function of a single variable, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$.
- $\zeta^*(y) = \sigma > 1$ under CES, $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$.
- Marshall's 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds iff $\zeta^{*\prime}(\cdot) < 0$.
- The choke price exists iff $\lim_{y\to 0} s^{*\prime}(y) < \infty$, which implies $\lim_{y\to 0} \frac{d\ln s^*(y)}{d\ln y} = 1$ and hence $\lim_{y\to 0} \zeta^*(y) = \infty$. For example, translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$.

Production Function: By integrating $= \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)$,

$$\frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left[\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega\right],$$

where

$$\Phi^*(y) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* = \frac{\int_0^y [s^*(\xi^*)/\xi^*]\, d\xi^*}{\int_0^y [s^*(y)/y]\, d\xi^*} > 1,$$

and $c^* > 0$ is a constant, proportional to TFP. $\Phi^*(y) > 1$ follows from $\mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$ implying that $s^*(y)/y$ is decreasing in y.

Notes:

- $X(\mathbf{x})$, linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $X(\mathbf{x})/A^*(\mathbf{x})$is not constant and depends on $\mathbf{x}$, with the sole exception of CES, because

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega {s^*}'(y_\omega)}{\int_\Omega {s^*}'(y_{\omega'}) y_{\omega'} d\omega'} = \frac{\left[1 - \frac{1}{\zeta^*(y_\omega)}\right] s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^*(y_{\omega'})}\right] s^*(y_{\omega'}) d\omega'} \neq \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^*(y)$ is independent of y or $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$ with $\zeta^*(y) = \sigma > 1$.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$1 = s^*\left(\frac{x}{A^*(x\mathbf{1}_\Omega)}\right)V = s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right)V \Longrightarrow y_\omega \equiv \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	Under H.S.A.
Price Elasticity	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta^*\left(\dfrac{x_\omega}{A^*(\mathbf{x})}\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^*\left(s^{*-1}(1/V)\right) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \dfrac{d \ln z(V)}{d \ln V} = -\dfrac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \Phi^*\left(s^{*-1}(1/V)\right) - 1 > 0.$

Notes:

- At the symmetric quantity patterns,

$$\ln\left[\frac{X(\mathbf{x})}{c^*A^*(\mathbf{x})}\right] = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) = \mathcal{L}(V) + 1.$$

- Since $s^{*-1}(1/V)$ is decreasing in V,

$$\sigma(V) = \zeta^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

implies that Marshall's 2nd law, $\zeta^{*\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$.

$$\sigma(V) = \zeta^*\left(s^{*-1}\left(\frac{1}{V}\right)\right); \ \mathcal{L}(V) = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1, \qquad \text{where} \quad \zeta^*(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1}; \ \Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^*.$$

Lemma 1*

$$\zeta^{*\prime}(y) \lesseqqgtr 0, \forall y \in (0, y_0) \Longrightarrow \Phi^{*\prime}(y) \gtreqqless 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{*\prime}(y) = 0 \iff \Phi^{*\prime}(y) = 0 \iff \text{CES}.$$

From this,

Proposition 1*

$$\zeta^{*\prime}(y) \lesseqqgtr 0, \forall y \in (0, y_0) \iff \sigma'(V) \gtreqqless 0, \forall V \in (1/s^*(y_0), \infty)$$

$$\Longrightarrow$$

$$\Phi^{*\prime}(y) \gtreqqless 0, \forall y \in (0, y_0) \iff \mathcal{L}'(V) \lesseqqgtr 0, \forall V \in (1/s^*(y_0), \infty)$$

Furthermore,

$$\zeta^{*\prime}(y) = 0 \iff \sigma'(V) = 0 \iff \Phi^{*\prime}(y) = 0 \iff \mathcal{L}'(V) = 0 \iff \text{CES}.$$

Thus, under H.S.A.,

- Marshall's 2nd Law, $\zeta^{*\prime}(\cdot) < 0$ for all $y > 0$ is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Equivalence of the Two Definitions of H.S.A.

Under the isomorphism given by the one-to-one mapping btw $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

From this,

$$\zeta^*(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} = \zeta(z) \equiv 1 - \frac{d\ln s(z)}{d\ln z} > 1,$$

$$0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1 \Leftrightarrow \mathcal{E}_s(z) \equiv \frac{d\ln s(z)}{d\ln z} < 0.$$

$y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

$$\frac{z_\omega \zeta'(z_\omega)}{y_\omega \zeta^{*\prime}(y_\omega)} = -\zeta(z_\omega) = -\zeta^*(y_\omega) < 0.$$

If $\lim_{y\to 0} s^{*\prime}(y) < \infty$, $\lim_{y\to 0} \zeta^*(y) = \infty$ and the (normalized) choke price is:

$$\lim_{y\to 0} \frac{s^*(y)}{y} = \lim_{y\to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega) d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right]$$

which is a constant iff CES.

Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right] d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \overline{z} \leftrightarrow \xi^* = 0,$$

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* - \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi = \frac{1}{s(z)}\int_{\bar{z}}^{z}\left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi - \frac{1}{s(z)}\int_z^{\bar{z}}\frac{s(\xi)}{\xi}d\xi = 1.$$

Since

$$w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}}[s(\xi')/\xi']\,d\xi'} \Leftrightarrow s(z)\Phi(z)w(\xi) = \frac{s(\xi)}{\xi}$$

$$w^*(\xi^*) \equiv \frac{s^*(\xi^*)/\xi^*}{\int_0^y \left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right]d\xi^{*\prime}} \Leftrightarrow s^*(y)\Phi^*(y)w^*(\xi^*) = \frac{s^*(\xi^*)}{\xi^*},$$

this implies

$$\frac{\xi w(\xi)}{\xi^* w^*(\xi^*)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\frac{c}{c^*} = \exp\left[\int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)]d\omega\right] = \exp\left[\int_\Omega s(z_\omega)d\omega\right] = e.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$